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Turing degrees of certain isomorphic images of computable relations

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This paper is dedicated to Chris Ash, who invented α-systems

Abstract

A model is computable if its domain is a computable set and its relations and functions are uniformly computable. Let $\mathscr{A}$ be a computable model and let R be an extra relation on the domain of $\mathscr{A}$. That is, R is not named in the language of $\mathscr{A}$. We define $Dg_{\mathscr{A}}(R)$ to be the set of Turing degrees of the images $f(R)$ under all isomorphisms f from $\mathscr{A}$ to computable models. We investigate conditions on $\mathscr{A}$ and R which are sufficient and necessary for $Dg_{\mathscr{A}}(R)$ to contain every Turing degree. These conditions imply that if every Turing degree $\leqslant \mathbf{0}''$ can be realized in $Dg_{\mathscr{A}}(R)$ via an isomorphism of the same Turing degree as its image of R, then $Dg_{\mathscr{A}}(R)$ contains every Turing degree. We also discuss an example of $\mathscr{A}$ and R whose $Dg_{\mathscr{A}}(R)$ coincides with the Turing degrees which are $\leqslant \mathbf{0}'$. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction and notation

We consider only computable first-order languages and only countable models. Models are denoted by script letters, and their domains by the corresponding capital Latin letters. The isomorphism of models is denoted by $\cong$. Let $\mathscr{A}$ be a model. $L(\mathscr{A})$ is the language of $\mathscr{A}$. $L(\mathscr{A})_A$ is the language $L(\mathscr{A}) \cup \{\mathbf{a}\colon a \in A\}$. $\mathscr{A}_A$ is the expansion of $\mathscr{A}$ to the language $L(\mathscr{A})_A$ such that every $\mathbf{a}$ is interpreted by a. A basic sentence is an atomic sentence or the negation of an atomic sentence. The atomic diagram of $\mathscr{A}$ is the set of all basic sentences of $L(\mathscr{A})_A$ which are true in $\mathscr{A}_A$. Let α be a computable ordinal. Ash [1] has defined computable Σ_α and Π_α formulae of $L_{\omega_1\omega}$, recursively and simultaneously, and together with their Gödel numbers (because the indexing of formulae in infinite disjunctions and conjunctions will be by their Gödel numbers).

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The computable Σ_0 and Π_0 formulae are the finitary quantifier-free formulae. The computable $\Sigma_{\alpha+1}$ ($\Pi_{\alpha+1}$, respectively) formulae are computably enumerable disjunctions (conjunctions, respectively) of $\exists\Pi_\alpha$ ($\forall\Sigma_\alpha$, respectively) formulae. If α is a limit ordinal, then the Σ_α (Π_α, respectively) formulae are of the form $\bigvee_{n\in W}\theta_n$ ($\bigwedge_{n\in W}\theta_n$, respectively), where W is a computably enumerable set of natural numbers and there is a sequence $(\alpha_n)_{n\in W}$ of ordinals having limit α, given by the ordinal notation for α, such that θ_n is a Σ_{α_n} (Π_{α_n}, respectively) formula. For a more precise definition of computable Σ_α and Π_α formulae, see [1]. A sequence of variables displayed after a formula contains all free variables occurring in the formula.

A model $\mathscr{A}$ is computable if its domain A is a computable set and the relations and functions of $\mathscr{A}$ are uniformly computable. Equivalently, $\mathscr{A}$ is a computable model if A is computable and the atomic diagram of $\mathscr{A}$ is computable. That is, A is computable and there is a computable enumeration $(a_i)_{i\in\omega}$ of A and an algorithm which determines for every quantifier-free formula $\theta(x_{i_0},\ldots,x_{i_{n-1}})$ in $L(\mathscr{A})$ and for every sequence $(a_{i_0},\ldots,a_{i_{n-1}})\in A^n$, whether $\mathscr{A}_A \models \theta(\mathbf{a}_{i_0},\ldots,\mathbf{a}_{i_{n-1}})$.

Let R be an additional relation on the domain of a computable model $\mathscr{A}$. That is, R is not named in $L(\mathscr{A})$. For simplicity, we assume that R is unary. (However, all definitions introduced and results established can be easily extended to relations of arbitrary arity.) For various computability-theoretic complexity classes $\mathscr{P}$, Ash and Nerode and others have investigated syntactic conditions on $\mathscr{A}$ and R under which for every isomorphism f from $\mathscr{A}$ onto a computable model $\mathscr{B}$, $f(R)\in\mathscr{P}$. Such relations R are called *intrinsically* $\mathscr{P}$ on $\mathscr{A}$. For example, Ash and Nerode [5] have established that, under some extra decidability condition on $\mathscr{A}$ (which involves R), R is intrinsically c.e. if and only if R is definable by a computable Σ_1 formula with finitely many parameters. Barker [6] has extended this result to every computable ordinal $\alpha \geqslant 2$. He has established that, under certain extra decidability conditions on $\mathscr{A}$, R is intrinsically Σ^0_α on $\mathscr{A}$ if and only if R is definable by a computable Σ_α formula with finitely many parameters. In the previous results, the extra decidability conditions are only needed to show that the corresponding syntactic conditions are necessary. We [8] have defined the *(Turing) degree spectrum* of R on $\mathscr{A}$, in symbols $Dg_{\mathscr{A}}(R)$, to be the set of all Turing degrees of the images of R under all isomorphisms from $\mathscr{A}$ onto computable models. For a computable model $\mathscr{B}$ such that $\mathscr{B}\cong\mathscr{A}$, the (Turing) degree spectrum of R on $\mathscr{A}$ with respect to $\mathscr{B}$, in symbols $Dg_{\mathscr{A},\mathscr{B}}(R)$, is the set of all Turing degrees of the images $f(R)\subseteq B$ under all isomorphisms f from $\mathscr{A}$ to $\mathscr{B}$. In [8] we have studied uncountable degree spectra, and have established conditions which are sufficient for $Dg_{\mathscr{A}}(R)$ to contain all Turing degrees. Here we prove that these conditions are necessary. For another independent proof, see [2].

The computability-theoretic notation is standard and as in [12]. We review some of it. By D_x we denote the finite set of natural numbers whose canonical index is x. Thus, $D_0=\emptyset$. If φ is a partial function, then $dom(\varphi)$ is the domain of φ, $rng(\varphi)$ is the range of φ, and $\varphi(a)\downarrow$ denotes that $a\in dom(\varphi)$. The concatenation of sequences is denoted by $\hat{}$. We often identify a set X with its characteristic function χ_X. We fix $\langle\cdot,\cdot\rangle$ to be a computable bijection from ω^2 onto ω. Let $X\subseteq\omega$. Then φ^X_0, φ^X_1,

$\varphi_2^X, \ldots$ is a fixed effective enumeration of all unary X-computable functions. φ_e^X is also denoted by $\{e\}^X$. We write $\varphi_{e,s}^X(n) = m$ if $e, n, m < s$, only numbers $z < s$ are used in the computation, and $\varphi_e^X(n) = m$ in fewer than s steps. Let $p \in 2^{<\omega}$. We write $\varphi_{e,s}^p(n) = m$ if $\varphi_{e,s}^X(n) = m$ for some $X \supset p$ and only elements in $dom(p)$ are used in the computation. Let $Y \subseteq \omega$. The join $X \oplus Y$ is $\{2n \colon n \in X\} \cup \{2n+1 \colon n \in Y\}$. By $X \leqslant_T Y$ ($X \equiv_T Y$, respectively) we denote that X is Turing reducible to Y (X is Turing equivalent to Y, respectively). $X <_T Y$ denotes that $X \leqslant_T Y$ but $Y \nleqslant_T X$. $\mathbf{x} = \deg(X)$ is the Turing degree of X. Hence $\mathbf{0} = \deg(\emptyset)$ and $\mathbf{x}^{(n)} = \deg(X^{(n)})$, where $X^{(n)}$ is the nth jump of X. A Turing degree is c.e. (Δ_2^0, respectively) if it contains a c.e. (Δ_2^0, respectively) set. The set of all Turing degrees is denoted by $\mathscr{D}$. A binary function $f : \omega^2 \to \omega$ is called selective if for every $x, y \in \omega$, $f(x, y) \in \{x, y\}$. X is a *semirecursive* set if there is a selective computable function such that if exactly one of x, y belongs to X, then $f(x, y)$ selects the element in X. An example of a semirecursive set is the deficiency set of a non-computable c.e. set for a 1–1 computable enumeration.

2. Realizing every Turing degree in a degree spectrum

Let $\mathscr{A}$ be a computable model and let R be an extra relation on the domain A of $\mathscr{A}$. As mentioned before, we will assume, without loss of generality, that R is unary. Let a computable model $\mathscr{B}$ be such that $\mathscr{A} \cong \mathscr{B}$. By $\mathscr{I}(\mathscr{A}, \mathscr{B})$ we denote the set of all isomorphisms from $\mathscr{A}$ to $\mathscr{B}$. We say that a partial function p from A to B is a *finite isomorphism* from $\mathscr{A}$ to $\mathscr{B}$ if p is 1–1, $dom(p)$ is finite and for every atomic formula $\alpha = \alpha(x_0, \ldots, x_{n-1})$ in $L(\mathscr{A})$, and every $a_0, \ldots, a_{n-1} \in dom(p)$, we have

$$\mathscr{A}_A \models \alpha(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}) \;\Leftrightarrow\; \mathscr{B}_B \models \alpha(\mathbf{b}_0, \ldots, \mathbf{b}_{n-1}).$$

where $b_0 = p(a_0), \ldots, b_{n-1} = p(a_{n-1})$. By $\mathscr{I}_{fin}(\mathscr{A}, \mathscr{B})$ we denote the set of all finite isomorphisms from $\mathscr{A}$ to $\mathscr{B}$. In [8] we have defined the R-equivalence relation $\sim_R$ on $\mathscr{I}_{fin}(\mathscr{A}, \mathscr{B})$ as follows:

$$q \sim_R r \;\Leftrightarrow\; (\forall b \in ran(q) \cap ran(r))[q^{-1}(b) \in R \Leftrightarrow r^{-1}(b) \in R].$$

Equivalently,

$$q \sim_R r \;\Leftrightarrow\; (\forall b \in ran(q) \cap ran(r))[b \in q(R) \Leftrightarrow b \in r(R)].$$

Since for every Turing degree $\mathbf{x}$, there are at most countably many Turing degrees which are $\leqslant \mathbf{x}$, and since every countable set of Turing degrees has an upper bound, a set of Turing degrees is uncountable if and only if it is unbounded.

Theorem 2.1 (Harizanov [8]). (i) *The following are equivalent*:

(0) $Dg_{\mathscr{A}}(R)$ *is uncountable.*

(1) $Dg_{\mathscr{A},\mathscr{B}}(R)$ *is uncountable.*

(2) $Dg_{\mathscr{A},\mathscr{B}}(R)$ *has cardinality* 2^{ω}.

(3) *There is a non-empty set* $\mathbb{S} \subseteq \mathscr{I}_{fin}(\mathscr{A}, \mathscr{B})$ *such that the following two conditions are satisfied:*

(A) $(\forall p \in \mathbb{S})(\forall a \in A)(\forall b \in B)(\exists q \in \mathbb{S})[q \supseteq p \wedge a \in dom(q) \wedge b \in ran(q)]$;

(B) $(\forall p \in \mathbb{S})(\exists q, r \in \mathbb{S})[q \supseteq p \wedge r \supseteq p \wedge \neg(q \sim_R r)]$.

(ii) *Let* $\mathbb{S}$ *be as in* (3). *Then for every set* $C \geqslant_T \mathbb{S}$, *there is an isomorphism* f *from* $\mathscr{A}$ *to* $\mathscr{B}$ *such that*

$$C \equiv_T f(R) \oplus \mathbb{S} \equiv_T f \oplus \mathbb{S}.$$

In particular, if $\mathbb{S}$ *is computable* (*or c.e.*), *then* $Dg_{\mathscr{A},\mathscr{B}}(R) = \mathscr{D}$ *and, moreover, for every set* $C \subseteq \omega$, *there is an isomorphism* f *from* $\mathscr{A}$ *to* $\mathscr{B}$ *such that*

$$C \equiv_T f(R) \equiv_T f.$$

In [8], we have also given examples of uncountable degree spectra $Dg_{\mathscr{A},\mathscr{B}}(R)$ such that $Dg_{\mathscr{A},\mathscr{B}}(R) \neq \mathscr{D}$. Now, we further investigate degree spectra which coincide with $\mathscr{D}$. The following example motivates the theorem that follows it.

Clearly, $\mathscr{Q} = (Q, \leqslant)$, where Q is the set of all rational numbers, is a computable model. $X \subseteq Q$ is an initial segment of $\mathscr{Q}$ if

$$(\forall a, b \in Q)[(a \in X \wedge b \leqslant a) \Rightarrow b \in X].$$

Example 2.1. Every Turing degree contains an initial segment of $\mathscr{Q}$. That is, if $R = \{q \in Q\colon q < \sqrt{2}\}$, then $Dg_{\mathscr{Q},\mathscr{Q}}(R) = \mathscr{D}$.

Proof. Let C be an arbitrary infinite coinfinite set of natural numbers. We will show that there is an initial segment X of $\mathscr{Q}$ of the same Turing degree as C. We define a real number r_C by

$$r_C = \sum_{n \in C} \frac{1}{2^n}.$$

Let X be the initial segment of $\mathscr{Q}$ determined by r_C. That is, $X = \{q \in Q\colon q < r_C\}$.

First, let us prove that $C \leqslant_T X$. By transfinite induction on k, we will show that we can X-computably determine whether $k \in C$. Assume that we can determine, computably in X, $C \cap \{0, \ldots, k-1\}$. Then we can find, computably in X, $\sum_{n \in C \cap \{0,\ldots,k-1\}} 1/2^n$. If $k \in C$, then, since C is infinite, $(\sum_{n \in C \cap \{0,\ldots,k-1\}} 1/2^n) + 1/2^k < r_C$. Conversely, if $(\sum_{n \in C \cap \{0,\ldots,k-1\}} 1/2^n) + (1/2^k) < r_C$, then, since C is coinfinite and $1/2^k = 1/2^{k+1} + 1/2^{k+2} + \cdots$, we conclude that $k \in C$. Hence,

$$k \in C \Leftrightarrow \left(\sum_{n \in C \cap \{0,\ldots,k-1\}} \frac{1}{2^n}\right) + \frac{1}{2^k} \in X.$$

Thus, we can determine, computably in X, whether $k \in C$.

Now, let us prove that $X \leqslant_T C$. We will establish the following equivalence:

$$q \in X \Leftrightarrow \exists n_0 \left[\sum_{n \in C \cap \{0,\dots,n_0\}} \frac{1}{2^n} \geqslant q \right].$$

The implication $\Leftarrow$ is clear. Conversely, if $\forall n_0[\sum_{n \in C \cap \{0,\dots,n_0\}} 1/2^n < q]$, then $r_C \leqslant q$, so $q \notin X$.

If $q > r_C$, then $\exists n_0[q - r_C > 1/2^{n_0}]$, hence $[q - \sum_{n \in C \cap \{0,\dots,n_0\}} 1/2^n] > 1/2^{n_0}$. Conversely, if $[q - \sum_{n \in C \cap \{0,\dots,n_0\}} 1/2^n] > 1/2^{n_0}$, then, since C is coinfinite, we conclude that $q - r_C > 0$. Therefore, for $q \neq r_C$,

$$q \notin X \Leftrightarrow \exists n_0 \left[q - \sum_{n \in C \cap \{0,\dots,n_0\}} \frac{1}{2^n} > \frac{1}{2^{n_0}} \right].$$

Hence, to decide for a given $q \in Q$, computably in C, whether $q \in X$, we search for n_0 such that either

$$\sum_{n \in C \cap \{0,\dots,n_0\}} \frac{1}{2^n} \geqslant q$$

or

$$\left[q - \sum_{n \in C \cap \{0,\dots,n_0\}} \frac{1}{2^n} \right] > \frac{1}{2^{n_0}}. \qquad \square$$

Theorem 2.2. *The following are equivalent*:

(1) $Dg_{\mathscr{A},\mathscr{B}}(R) = \mathscr{D}$ *and, moreover, for every set* $C \subseteq \omega$, *there is an isomorphism* f *from* $\mathscr{A}$ *to* $\mathscr{B}$ *such that* $C \equiv_T f(R) \equiv_T f$.

(2) *There is* $e \in \omega$ *and* $p \in 2^{<\omega}$ *such that the set*

$$\mathbb{S}_{e,p} =_{\text{def}} \{\varphi_e^q \colon q \in 2^{<\omega} \wedge q \supseteq p\}$$

has the following properties:

$$\mathbb{S}_{e,p} \subseteq \mathscr{I}_{fin}(\mathscr{A},\mathscr{B}),$$

(A) *from Theorem* 2.1 *is satisfied for* $\mathbb{S} = \mathbb{S}_{e,p}$, *and*

$$(\exists i \in \omega)(\forall q \supseteq p)(\forall a \in dom(q))[\varphi_i^{\varphi_e^q(R)}(a) \downarrow = q(a)].$$

(3) *There is a non-empty computable* (*or c.e.*) *set* $\mathbb{S} \subseteq \mathscr{I}_{fin}(\mathscr{A},\mathscr{B})$ *such that the conditions* (A) *and* (B) *from Theorem* 2.1 *are satisfied.*

Proof. $\neg(2) \Rightarrow \neg(1)$ Assume the negation of (2). That is, for every $\langle e, i\rangle$ and every $p = 2^{<\omega}$, there is $q \in 2^{<\omega}$ such that $q \supseteq p$ and

(i) $\varphi_e^q \notin \mathscr{I}_{fin}(\mathscr{A},\mathscr{B})$ or

(ii) $(\exists a \in A)(\forall r \supseteq q)[a \notin dom(\varphi_e^r)]$ or

(iii) $(\exists b \in B)(\forall r \supseteq q)[b \notin ran(\varphi_e^r)]$ or

(iv) $(\exists a \in dom(q))[\varphi_i^{\varphi_e^q(R)}(a)\downarrow \neq q(a)]$.

We will now use a finite extension argument to construct the characteristic function of a set $C \subseteq \omega$ which satisfies the following requirement for every $\langle e,i \rangle$:

$$R_{\langle e,i \rangle}: \varphi_e^C \in \mathscr{I}(\mathscr{A},\mathscr{B}) \Rightarrow \varphi_i^{\varphi_e^C(R)} \neq C.$$

Construction Let $p_{-1} =_{\text{def}} \emptyset$.

Stage s. Let $s = \langle e,i \rangle$. We have already constructed $p_{s-1} \in 2^{<\omega}$. Let q be the least binary sequence such that $q \supseteq p_{s-1}$ and one of the conditions (i)–(iv) is satisfied. Let $p_s =_{\text{def}} q$. End of construction.

Let $C \subseteq \omega$ be such that $\chi_C = \bigcup_{s \geqslant -1} p_s$. Hence, for $f \in \mathscr{I}(\mathscr{A},\mathscr{B})$, if $f \leqslant_T C$, that is, if $f = \varphi_e^C$ for some $e \in \omega$, then $\neg(C \leqslant_T f(R))$. Let $\mathbf{c} = deg(C)$. Thus, $\mathbf{c}$ cannot be realized in $Dg_{\mathscr{A},\mathscr{B}}(R)$ via an isomorphism of degree $\mathbf{c}$.

(2)$\Rightarrow$(3) Fix the corresponding e and p. By assumption, $\mathbb{S}_{e,p} \subseteq \mathscr{I}_{fin}(\mathscr{A},\mathscr{B})$ and (A) is satisfied for $\mathbb{S} = \mathbb{S}_{e,p}$. Let us show that (B) is also satisfied for $\mathbb{S} = \mathbb{S}_{e,p}$. Fix the corresponding $i \in \omega$. Let $p_1 \in 2^{<\omega}$ be such that $p_1 \supseteq p$. Now, choose binary sequences q and r such that $q \supseteq p_1$, $r \supseteq p_1$, and

$$(\exists a \in dom(q) \cap dom(r))[q(a) \neq r(a)].$$

Then

$$\varphi_i^{\varphi_e^q(R)}(a)\downarrow \neq \varphi_i^{\varphi_e^r(R)}(a)\downarrow.$$

Hence,

$$\exists b[b \in \varphi_e^q(R) \Leftrightarrow b \notin \varphi_e^r(R)].$$

Thus, $\neg(\varphi_k^q \sim_R \varphi_k^r)$.

(3)$\Rightarrow$(1) This is already proven in [8] (see (ii) of Theorem 2.1). $\square$

The equivalence of (1) and (3) in Theorem 2.2 has also been established independently by Ash et al. in [2]. Their proof uses the forcing method.

Remark 2.1. In the proof of $\neg$(2)$\Rightarrow\neg$(1) for Theorem 2.2, the construction of C can be done computably in $\emptyset''$. Hence $C \in \Delta_3^0$. Thus, if not every Turing degree is obtained in a degree spectrum $Dg_{\mathscr{A},\mathscr{B}}(R)$ via an isomorphism of the same Turing degree, then there is such a Δ_3^0 degree. This conclusion also follows from the proof in [2] since there is a 2-generic Δ_3^0 set.

3. Realizing Δ_2^0 degrees in a degree spectrum

In [9] we have given a general condition for $\mathscr{A}$ and R which is sufficient for every c.e. degree to be realized in $Dg_{\mathscr{A}}(R)$ via a c.e. set of the same Turing degree as the corresponding isomorphism. This condition is satisfied by the following model $\mathscr{A}_0$ and relation R_0.

Let $\mathscr{A}_0 = (\omega, \prec)$ be the following computable linear order of order type $\omega + \omega^*$:

$$0 \prec 2 \prec 4 \prec \cdots \prec 5 \prec 3 \prec 1.$$

A computable relation R_0 is the initial segment of type ω; that is, $R_0 = 2\omega$.

Hence, every c.e. degree can be realised in $Dg_{\mathscr{A}_0}(R_0)$ via a c.e. set of the same Turing degree as the corresponding isomorphism. It is easy to see that R_0 is intrinsically Δ^0_2 on $\mathscr{A}$, because it satisfies the syntactic condition in [6]. Namely,

$$x \in R_0 \;\Leftrightarrow\; \bigvee_{n \in \omega} \exists x_0 \ldots \exists x_n [x_0 \prec x_1 \prec \cdots \prec x_n \wedge x = x_n \wedge$$
$$\forall y[\neg(y \prec x_0) \wedge \neg(x_0 \prec y \prec x_1) \wedge \cdots \wedge \neg(x_{n-1} \prec y \prec x_n)]],$$

and

$$x \notin R_0 \;\Leftrightarrow\; \bigvee_{n \in \omega} \exists x_0 \ldots \exists x_n [x_0 \succ x_1 \succ \cdots \succ x_n \wedge x = x_n \wedge$$
$$\forall y[\neg(y \succ x_0) \wedge \neg(x_0 \succ y \succ x_1) \wedge \cdots \wedge \neg(x_{n-1} \succ y \succ x_n)]].$$

Ash et al. [2] have extended the sufficient condition in [9] to the αth level in Ershov's classification of Δ^0_2 degrees, where α is any fixed computable ordinal. A Turing degree is α-c.e. if it contains an α-c.e. set. A set $C \subseteq \omega$ is α-c.e. if there is a computable function $f : \omega^2 \to \{0, 1\}$ and a computable function $o : \omega^2 \to \{\beta : \beta \text{ is an ordinal} \wedge \beta \leqslant \alpha\}$ with the following properties:

$$(\forall x)\left[\lim_{s \to \infty} f(x,s) = C(x) \wedge f(x,0) = 0\right],$$
$$(\forall x)(\forall s)[o(x,s+1) \leqslant o(x,s) \wedge o(x,0) = \alpha],$$

and

$$(\forall x)(\forall s)[f(x,s+1) \neq f(x,s) \Rightarrow o(x,s+1) < o(x,s)].$$

In particular, 1-c.e. sets are c.e. sets, and 2-c.e. sets are d-c.e. sets. For other equivalent definitions of α-c.e. sets, see [7, 4]. Epstein et al. [7] have shown that some levels in Ershov's hierarchy are notation-dependent, and that for every Δ^0_2 set X, there is an ordinal notation in which X is ω^2-c.e. Ash and Knight [4] have given a syntactic condition which is, under appropriate decidability conditions, sufficient and necessary for R to be intrinsically α-c.e. on $\mathscr{A}$. As a corollary, they have shown that for every computable ordinal α, R_0 is not intrinsically α-c.e. on $\mathscr{A}_0$. This result also follows from the following proposition because for a fixed ordinal notation, the α-c.e. degrees form a proper hierarchy (see [7, Theorem 9]).

Proposition 3.1. *$Dg_{\mathscr{A}_0}(R_0)$ consists of all Δ^0_2 degrees.*

Proof (1). Jockusch [11, Theorem 5.2] has established that every non-zero Turing degree computable in $\mathbf{0}'$ contains a semirecursive set which is both immune and coimmune. However, a set of natural numbers is semirecursive if and only if it is an initial segment of a computable linear ordering on ω (see [11, Theorem 4.1]). Let $\mathbf{c}$ be an arbitrary non-zero Δ_2^0 degree. Hence, there is a computable linear ordering $\mathscr{B}=(\omega,\prec_{\mathscr{B}})$ and an initial segment X on $\mathscr{B}$ such that $deg(X)=\mathbf{c}$ and X is immune and coimmune. Since X is immune, no element of X can have infinitely many predecessors. Similarly, no element of $\omega - X$ can have infinitely many successors. Thus, the order type of $\mathscr{B}$ is $\omega+\omega^*$, and X is the ω-part of $\mathscr{B}$. In other words, there is an isomorphism f from $\mathscr{A}_0$ to $\mathscr{B}$ such that $f(R_0)=X$. Therefore, we conclude that $DgSp_{\mathscr{A}_0}(R_0)$ is the set of all Δ_2^0 degrees. □

We will also give a direct proof by constructing a computable model $\mathscr{B}$ isomorhic to $\mathscr{A}_0$ and a corresponding isomorphism. In the proof, we will consider binary trees. Such trees can be viewed as growing downward from the top node $\emptyset$. Let $\nu,\mu\in 2^{<\omega}$. As usual, we say that ν is to the left of μ, in symbols $\nu <_L \mu$, if

$$\exists\gamma\in 2^{<\omega}[\gamma\hat{\ }0\subseteq\nu\wedge\gamma\hat{\ }1\subseteq\mu].$$

We have the following partial ordering on $2^{<\omega}$:

$$\nu<\mu \ \Leftrightarrow_{\ def}(\nu<_L\mu\vee\nu\subsetneq\mu).$$

Let $C\subseteq\omega$. We write $\nu<_L C$ if for $\gamma=C(0)\hat{\ }C(1)\hat{\ }\cdots\hat{\ }C(lh(\nu)-1)$, we have $\nu<_L\gamma$. We similarly define $C<_L\nu$ and $\nu<C$. Let r_C be defined as in Example 2.1. Notice that if C is infinite and coinfinite then $(\forall x\in\omega)$ $[\sum_{n\in D_x}1/2^n\neq r_C]$. Jockusch [11] has defined an infinite and coinfinite set $C\subseteq\omega$ to be *strongly non-c.e.* if neither the set $\{x\in\omega\colon \sum_{n\in D_x}1/2^n<r_C\}$ is c.e. nor the set $\{x\in\omega\colon \sum_{n\in D_x}1/2^n>r_C\}$ is c.e. He [11] has established that every non-zero Turing degree contains a strongly non-c.e. set.

Let $p\in A^m$ for some $m\in\omega$, and let let $\alpha=\alpha(x_0,\ldots,x_{m-1})$ be a formula. We say that p satisfies α in $\mathscr{A}$ if

$$\mathscr{A}\models\alpha(x_0,\ldots,x_{m-1})[p(0),\ldots,p(m-1)].$$

Proof (2). We will construct a computable model $\mathscr{B}$ isomorphic to $\mathscr{A}_0$. Let the domain B be ω. Let $\mathbf{c}$ be a non-zero Δ_2^0 degree. We choose a strongly non-c.e. set $C\subseteq\omega$ such that $deg(C)=\mathbf{c}$. Let $h:\omega^2\to\{0,1\}$ be a computable function which approximates C, that is,

$$(\forall n\in\omega)\left[C(n)=\lim_{s\to\infty}h(n,s)\right].$$

Now, we define the following computable binary tree:

$$T=\{h(0,s)\hat{\ }h(1,s)\hat{\ }\cdots\hat{\ }h(n,s)\colon n\leqslant s\wedge s\in\omega\}\cup\{\emptyset\}.$$

For every $s\in\omega$, T has the following node of length $s+1$:

$$\nu_s=h(0,s)\hat{\ }h(1,s)\hat{\ }\cdots\hat{\ }h(s,s).$$

At every stage s of the construction, we define a finite isomorphism $p_s : \{0, 1, \ldots, s\} \to A_0$. The function p_s has the following properties (*):

$$(\forall n \in \omega)[(2n+2 \in ran(p_s) \Rightarrow 2n \in ran(p_s)) \wedge (2n+1 \in ran(p_s) \Rightarrow 2n-1 \in ran(p_s))],$$
$$(\forall n, m \in \{0, 1, \ldots, s-1\})[v_n < v_m \Rightarrow p_s(n) \prec p_s(m)],$$

and

$$(\forall n \in \{0, 1, \ldots, s-1\})[(v_n < v_s \Rightarrow p_s(n) \in R_0) \wedge (v_s <_L v_n \Rightarrow p_s(n) \in \overline{R}_0)].$$

Construction

Stage 0. Let $p_0 =_{\mathrm{def}} \{(0, a)\}$, where a is the least element in R_0 if $v_0 < v_1$, and the least element in $\overline{R}_0$ if $v_1 <_L v_0$.

Stage $s > 0$. We have $p_{s-1} : \{0, 1, \ldots, s-1\} \to A_0$, satisfying the above properties (*), and a finite part $\mathscr{B}_{s-1}$ of the atomic diagram of $\mathscr{B}$, which involves constants $0, 1, \ldots, s-1$ and is determined by p_{s-1} and $\mathscr{A}_0$.

Let $n < s-1$ be the least number (if it exists, otherwise let $q =_{\mathrm{def}} p_{s-1}$) such that $v_s <_L v_n < v_{s-1}$ or $v_{s-1} <_L v_n < v_s$. We change p_{s-1} into the corresponding q with the same domian as p_{s-1} such that $(\forall m < n)[q(m) = p_{s-1}(m)]$, q preserves $\mathscr{B}_{s-1}$, and satisfies conditions (*). Let

$$p_s = q \cup \{(s, a)\},$$

Where a is the least element in $R_0 - ran(q)$ if $v_{s-1} < v_s$, and a the least element in $\overline{R}_0 - ran(q)$ if $v_s <_L v_{s-1}$.

Let $\mathscr{B}_s$ be the set of all basic sentences with Gődel number $\leqslant s$, involving constants $0, 1, \ldots, s$, which are satisfied by p_s in $\mathscr{A}_0$. Note that $\mathscr{B}_{s-1} \subseteq \mathscr{B}_s$. End of the construction.

Let the atomic diagram of $\mathscr{B}$ be $\bigcup_{s \geqslant 0} \mathscr{B}_s$. Thus, $\mathscr{B}$ is a computable model. Fix $n \in \omega$. Let s_n be the least number such that $s_n \geqslant n$ and

$$(\forall m \leqslant n)(\forall s \geqslant s_n)[h(m, s) = h(m, s_n) = C(m)].$$

Hence,

$$(\forall s \geqslant s_n)[p_s(n) = p_{s_n}(n)].$$

We define

$$f(n) = p_{s_n}(n).$$

f is a 1-1 function from B to A_0.

Lemma 3.2. *f is onto A_0.*

Proof. Assume inductively that $0, 1, \ldots, j-1 \in ran(f)$. We will prove that $j \in ran(f)$. Let $f(n_i) = i$ for $i < j$. Let $n = \max\{n_0, n_1, \ldots, n_{j-1}\}$ and let $t_0 = s_n$. Hence for every $s \geqslant t_0$, v_s extends $C(0)\hat{}C(1)\hat{}\cdots\hat{}C(n)$.

Case: $j \in R$. We claim that there exists $s' \geqslant t_0$ such that $(\forall s > s')[v_{s'} < v_s]$. Otherwise, we can effectively enumerate an infinite sequence $t_0 < t_1 < t_2 < \cdots$ such that for every $i \in \omega$, $v_{t_{i+1}} <_L v_{t_i}$. Since h approximates C, we conclude that $(\forall i \in \omega)[C <_L v_{t_i}]$. Hence for every $x \in \omega$,

$$\left(\sum_{n \in D_x} 1/2^n > r_C \right) \Leftrightarrow (\exists i \in \omega)[\chi_{D_x} \geqslant v_{t_i}].$$

Thus, the set $\{x \in \omega\colon \sum_{n \in D_x} \frac{1}{2^n} > r_C\}$ is c.e., contradicting the fact that C is strongly non-c.e.

We now choose the least stage s' with the property described above. It follows from the construction that $j \in ran(p_{s'+1})$ and that

$$(\forall s > s' + 1)[p_s^{-1}(j) = p_{s'+1}^{-1}(j)].$$

Hence $a_j \in ran(f)$.

Case: $j \in \overline{R}_0$.

As in the previous case, we prove that there exists $s' \geqslant t_0$ such that $(\forall s \geqslant s')[v_s <_L v_{s'}]$. For the least such s', it follows from the construction that

$$(\forall s > s' + 1)[p_s^{-1}(j) = p_{s'+1}^{-1}(j)].$$

Hence $j \in ran(f)$. $\square$

Lemma 3.3. $f^{-1}(R_0) \equiv_T C$

Proof. Let $X = f^{-1}(R_0)$. It follows by construction that

$$X = \{n \in \omega\colon v_n < C\}.$$

Hence,

$$X \leqslant_T C.$$

We now prove, by induction, that $C \leqslant_T X$. To determine whether $k \in C$, we assume that we can find σ using oracle X, where

$$\sigma = C(0)\hat{\ }C(1)\hat{\ }\cdots\hat{\ }C(k-1).$$

Then

$$k \in C \Leftrightarrow (\exists n \in X)[\sigma\hat{\ }(1) \subseteq v_n].$$

Equivalently,

$$k \notin C \Leftrightarrow (\exists n \in \overline{X})[\sigma\hat{\ }(0) \subseteq v_n]. \quad \square$$

Hird [10] has shown that there is a computable copy of $\mathscr{A}_0$ in which the initial segment of type ω is h-simple. However, Jim Owings (unpublished) has observed that every deficiency set of a non-computable c.e. set for a 1-1 computable enumeration is the initial segment of type ω of some computable linear order isomorphic to $\mathscr{A}_0$. That is because every such deficiency set is semirecursive and coimmune. Hence, for every c.e. non-computable set C, there is a computable copy of $\mathscr{A}_0$ in which the initial segment of type ω is h-simple and Turing equivalent to C. This conclusion has also been obtained for simple initial segments by Ash et al. in [3], as an example of their general result for the so-called quasi-simple relations on computable models. These simple sets are automatically h-simple because semirecursive immune sets are h-immune. On the other hand, such sets cannot be hh-simple because no semirecursive set can be hh-immune (see [11]). Hird [10] has also established that no interval of a computable linear order is hh-immune.

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